

On Gravitational Radiation in Quadratic $f(R)$ Gravity

Joachim N  f* and Philippe Jetzer

*Institut f  r Theoretische Physik, Universit  t Z  rich,
Winterthurerstrasse 190, CH-8057 Z  rich, Switzerland.*

Abstract

We investigate the gravitational radiation emitted by an isolated system for gravity theories with Lagrange density $f(R) = R + aR^2$. As a formal result we obtain leading order corrections to the quadrupole formula in General Relativity. We make use of the analogy of $f(R)$ theories with scalar-tensor theories, which in contrast to General Relativity feature an additional scalar degree of freedom. Unlike General Relativity, where the leading order gravitational radiation is produced by quadrupole moments, the additional degree of freedom predicts gravitational radiation of all multipoles, in particular monopoles and dipoles, as this is the case for the most alternative gravity theories known today. An application to a hypothetical binary pulsar moving in a circular orbit yields the rough limit $a \lesssim 1.7 \cdot 10^{17} \text{ m}^2$ by constraining the dipole power to account at most for 1% of the quadrupole power as predicted by General Relativity.

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*Electronic address: naef@physik.uzh.ch

I. INTRODUCTION

One of the most impressive endorsements of General Relativity Theory (GR) is the agreement of the predictions of the famous quadrupole formula for gravitational radiation with indirect measurements of the energy loss of binary pulsars. It is thus natural to test modified gravity theories by deriving the corrections to the quadrupole formula and comparing them with experimental data. For many types of theories this has been done in the past [1]. Though this problem is still open for metric $f(R)$ theories with an action

$$S = \frac{c^3}{16\pi G} \int f(R) \sqrt{-g} d^4x + S_M, \quad (1)$$

where in contrast to GR the Einstein–Hilbert Lagrangian density is replaced by a nonlinear function $f(R)$. We will always assume the function f to be smooth. S_M is the standard matter action. In the past years, this type of theories has become very popular to heuristically gain insight in the problem of dark energy. For an overview one may consult e. g. [2, 3] and references therein. It is also worth mentioning that quadratic corrections aR^2 to the Einstein–Hilbert action have been considered in the context of inflationary cosmology already in the 1980’s [4, 5].

In this article we prepare the way to investigate the energy emission of binary systems by gravitational radiation. The basic equations of $f(R)$ gravity are given in Section II. For our purposes it will be convenient to work in the scalar tensor formulation of quadratic $f(R)$ gravity. In Section III we employ the linearised field equations of quadratic $f(R)$ gravity to derive the weak gravitational fields emitted by a localised source and expand them into multipoles. The linearised $f(R)$ gravity has been investigated for example in [2, 6–9]. For a treatment of linearised scalar tensor theories see [1, 10]. In Section IV we dwell on the energy–momentum complex in quadratic $f(R)$ gravity as an analogue to the Landau–Lifshitz complex in GR. The leading order correction to the quadrupole formula in terms of momenta of the energy–momentum tensor is derived in Section V. In Section VI we finally illustrate the correction with an application to binary systems in circular orbits.

Notational conventions: Greek letters denote space time indices and range from 0 to 3, whereas Latin letters denote space indices and range from 1 to 3. We take the sum over repeated indices within a term.

II. THE FIELD EQUATIONS

Consider a 4-dimensional pseudo Riemannian manifold with metric $g_{\mu\nu}$ of signature $(-, +, +, +)$. We write $g = \det g_{\mu\nu}$ and denote the Ricci tensor of $g_{\mu\nu}$ by $R_{\mu\nu}$. The variation of the action (1) with respect to the metric yields the Euler–Lagrange equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square_g f'(R) = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (2)$$

where $R = g^{\mu\nu}R_{\mu\nu}$, $T_{\mu\nu} = (-2c/\sqrt{-g})(\delta S_M/\delta g^{\mu\nu})$ is the energy-momentum tensor, c the vacuum speed of light, G Newton’s constant, ∇_μ the covariant derivative for $g_{\mu\nu}$ and $\square_g = \nabla^\mu \nabla_\mu$. Taking the trace of (2) we obtain

$$3\square_g f'(R) + f'(R)R - 2f(R) = \frac{8\pi G}{c^4} T, \quad (3)$$

where T is the trace of $T_{\mu\nu}$. We now assume

$$f(R) = R + aR^2 \quad (4)$$

and make use of the equivalence between $f(R)$ gravity and scalar tensor theory by defining the scalar field $\phi := f'(R)$. This identification is feasible since $f''(R) \neq 0$ holds for our choice of $f(R)$, and $f'(R)$ is thus invertible. We define the scalar field φ by $\phi = 1 + 2a\varphi$, where we have chosen the asymptotic value such that a renormalisation of Newton’s constant is redundant. Then the equations (2) and (3) are equivalent to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{1+2a\varphi} \left[\frac{8\pi G}{c^4} T_{\mu\nu} + a \left(2\nabla_\mu \nabla_\nu \varphi - 2g_{\mu\nu} \square_g \varphi - \frac{1}{2}g_{\mu\nu} \varphi^2 \right) \right] \quad (5)$$

$$\square_g \varphi = \frac{4\pi G}{3ac^4} T + \frac{1}{6a} \varphi. \quad (6)$$

The field φ thus has the effective mass $\hbar/(c\sqrt{6a})$. Since we aim at the investigation of isolated systems at the scale of the Solar System, we expect the theory derived from (4) to be a small perturbation of General Relativity. Thus the dimensionless quantity aR should be small compared to 1. This still holds if the parameter a varies appropriately with the scalar curvature and therefore the local matter density. For our quadratic model this fact reflects the concept of the chameleon effect [11], which states the possibility that the Compton wave length $\lambda = \sqrt{6a}$ of the field φ is smaller or larger in regions with higher or lower matter density, respectively. In our specific case however, we assume the Compton wave length to be constant. In this sense theory should be viewed as a local effective field theory which is valid for a certain range of the parameters.

III. GRAVITATIONAL RADIATION IN $f(R)$ GRAVITY

Consider weak perturbations of the Minkowski spacetime metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (7)$$

where the coefficients of the perturbation satisfy $|h_{\mu\nu}| \ll 1$. In what follows the indices are raised and lowered by $\eta_{\mu\nu}$. For the field ϕ we have already chosen the asymptotic value 1, of which $2a\varphi$ is the perturbation. Moreover, the field equation (6) is inhomogeneous linear in φ , so that the linearisation in the perturbations is simply achieved through the replacement of \square_g by \square_η . Let $h = h_\mu^\mu$, define

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} - 2a\varphi\eta_{\mu\nu} \quad (8)$$

and choose the gauge

$$\gamma_{\mu\nu}{}^\nu = 0. \quad (9)$$

Up to linear order in $h_{\mu\nu}$ and φ , equation (5) can then be written as

$$\square_\eta \gamma_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}. \quad (10)$$

In the slow motion approximation and at large distances from the localised sources, a special solution of (10) can be derived in analogy with the GR case. Since $T_{\mu\nu}$ is divergence free, we can express the spatial components of $\gamma_{\mu\nu}$ in terms of the quadrupole momenta of T^{00} . Thus we obtain

$$\gamma^{ij}(t, \mathbf{x}) = \frac{2G}{c^6} \frac{1}{|\mathbf{x}|} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^3} d^3x' T^{00}(t - |\mathbf{x}|/c, \mathbf{x}') x'^i x'^j. \quad (11)$$

For the field φ we write equation (6) as

$$\square_\eta \varphi - \alpha^2 \varphi = \frac{8\pi G \alpha^2}{c^4} S, \quad (12)$$

where $\alpha := 1/\sqrt{6a}$ and

$$S = T \left[1 + \frac{1}{c^2} \left(3W + \frac{2}{3\alpha^2} V \right) \right] + \frac{1}{8\pi G} \left[\frac{1}{3\alpha^4} (\nabla V)^2 + UV + VW \right] \quad (13)$$

is the source T (to leading order) extended by the terms which are quadratic in the perturbations. These are expressed in terms of the Newtonian and post Newtonian potentials U ,

W and V , which are given by [12, 13]

$$\begin{aligned} U(\mathbf{x}, t) &= \frac{4G}{3c^2} \int \frac{(-2)T^{00}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \\ W(\mathbf{x}, t) &= \frac{2G}{3c^2} \int \frac{(-2)T^{00}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \left(1 - e^{-\alpha|\mathbf{x} - \mathbf{x}'|}\right) d^3x', \\ V(\mathbf{x}, t) &= \frac{2G\alpha^2}{c^2} \int \frac{(-2)T^{00}(\mathbf{x}', t)e^{-\alpha|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \end{aligned} \quad (14)$$

Here $(-2)T^{00}$ is the leading order time-time component of $T^{\mu\nu}$.

A special solution of (12) is the convolution of the source with the Green's function of the Klein-Gordon equation [14],

$$\mathcal{G}(t, \mathbf{x}) = -\frac{1}{4\pi} \left[\frac{\delta(t - |\mathbf{x}|/c)}{|\mathbf{x}|} - \frac{\alpha J_1\left(\alpha c \sqrt{t^2 - (|\mathbf{x}|/c)^2}\right) \theta(t - |\mathbf{x}|/c)}{\sqrt{t^2 - (|\mathbf{x}|/c)^2}} \right], \quad (15)$$

where J_1 the Bessel function of first order and the Dirac, and Heaviside, distribution kernel is denoted by δ , and θ , respectively. Then

$$\begin{aligned} \varphi(t, \mathbf{x}) &= -\frac{2G\alpha^2}{c^4} \left[\int_{\mathbb{R}^3} d^3x' \frac{S(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')} {|\mathbf{x} - \mathbf{x}'|} \right. \\ &\quad \left. - \int_{-\infty}^{t-|\mathbf{x}-\mathbf{x}'|/c} dt' \int_{\mathbb{R}^3} d^3x' \frac{\alpha J_1\left(\alpha c \sqrt{(t - t')^2 - (|\mathbf{x} - \mathbf{x}'|/c)^2}\right)}{\sqrt{(t - t')^2 - (|\mathbf{x} - \mathbf{x}'|/c)^2}} S(t, \mathbf{x}') \right]. \end{aligned} \quad (16)$$

If the source emits a single pulse, the field φ observed at a distance $|\mathbf{x}|$ consists of this pulse diminished by the factor $1/|\mathbf{x}|$ given by the first term on the right hand side of (18), and a wake represented by the second term. The scalar mode is thus dispersive; for a discussion of the dispersion of plane scalar waves see [6, 9, 10].

After the substitution

$$t' = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \sqrt{1 + \frac{s^2}{\alpha^2 |\mathbf{x} - \mathbf{x}'|^2}}, \quad dt' = -\frac{1}{\alpha^2 c |\mathbf{x} - \mathbf{x}'|} \frac{s}{\sqrt{1 + \frac{s^2}{\alpha^2 |\mathbf{x} - \mathbf{x}'|^2}}} ds \quad (17)$$

in the second term of the right hand side of (16), the solution can be written as

$$\varphi(t, \mathbf{x}) = \frac{G}{3ac^4} \int_{\mathbb{R}} ds [J_1(s)\theta(s) - \delta(s)] \int_{\mathbb{R}^3} d^3x' \frac{S\left(t - |\mathbf{x} - \mathbf{x}'| \sqrt{1 + \frac{6as^2}{|\mathbf{x} - \mathbf{x}'|^2}}/c, \mathbf{x}'\right)}{|\mathbf{x} - \mathbf{x}'| \sqrt{1 + \frac{6as^2}{|\mathbf{x} - \mathbf{x}'|^2}}}. \quad (18)$$

We assume the source S to be a smooth function of time. In the limit $a \rightarrow 0$, the spatial integral in (18) is independent of s , and because of $\int_0^\infty J_1(s)ds = 1$ we have $\lim_{a \rightarrow 0} a\varphi = 0$.

Let $r := |\mathbf{x}|$ and $\mathbf{n} := \mathbf{x}/r$. At large distances r from isolated and slowly moving sources, we can expand (18) into multipoles. Since the scalar contribution to the energy flux (equation (36) below) is quadratic in φ and $\gamma_{\mu\nu}$, and moreover, the leading order radiation predicted by GR is produced by quadrupole moments, we need an expansion for φ taking into account up to hexadecapole moments. We therefore derive the fourth order Taylor polynom of the integrand in (18) around the origin in the variable \mathbf{x}' , while ignoring the explicit dependence of S on \mathbf{x}' . Consider a source with an extension characterised by a typical length $d \ll r$ and moving slowly at a velocity characterised by a typical frequency $\omega \ll c/d$. The polynomial can be written as a sum of terms of approximate order $\mathcal{O}\left(\frac{d^n}{r^m c^l} \frac{\partial^l}{\partial t^l} S\right) = \mathcal{O}\left(\frac{d^n \omega^l}{r^m c^l}\right)$, $m + l = n$. In fact the expansion coefficients of the monomials $x'^{i_1} \cdots x'^{i_n}$ are not proportional to r^{-m} , but depend on r also by means of the function

$$p(s) := \left(1 + \frac{6as^2}{r^2}\right)^{-1/2}. \quad (19)$$

For convenience, we will ignore the function p when making use of the Landau Symbol \mathcal{O} . We define the retarded time

$$\tau(t, s) := t - \frac{r}{p(s)c} \quad (20)$$

and the distribution kernel

$$q(s) := p(s)[J_1(s)\theta(s) - \delta(s)]. \quad (21)$$

We are then left with

$$\begin{aligned} \varphi(t, \mathbf{x}) = & \frac{G}{3ac^4 r} \int_{\mathbb{R}} ds q(s) \int_{\mathbb{R}^3} d^3 x' \left[1 + F_i(s)x'^i + F_{ij}(s)x'^i x'^j \right. \\ & \left. + F_{ijk}(s)x'^i x'^j x'^k + F_{ijkl}(s)x'^i x'^j x'^k x'^l \right] S(\tau(t, s), \mathbf{x}') + \mathcal{O}\left(\frac{d^5 \omega^m}{r^n c^m}\right), \end{aligned} \quad (22)$$

$(n + m = 5)$, where

$$\begin{aligned}
F_i(s) &:= n_i \left[\frac{p^2(s)}{r} + \frac{p(s)}{c} \frac{\partial}{\partial t} \right], \\
F_{ij}(s) &:= n_i n_j \left[\frac{3p^4(s)}{2r^2} + \frac{p^3(s)}{rc} \frac{\partial}{\partial t} + \frac{p^2(s)}{2c^2} \frac{\partial^2}{\partial t^2} \right] - \delta_{ij} \left[\frac{p^2(s)}{2r^2} + \frac{p(s)}{2rc} \frac{\partial}{\partial t} \right], \\
F_{ijk}(s) &:= n_i n_j n_k \left[\frac{5p^6(s)}{2r^3} + \frac{p^5(s)}{2r^2 c} \frac{\partial}{\partial t} + \frac{p^4(s)}{2r c^2} \frac{\partial^2}{\partial t^2} + \frac{p^3(s)}{6c^3} \frac{\partial^3}{\partial t^3} \right] \\
&\quad - n_i \delta_{jk} \left[\frac{3p^4(s)}{2r^3} + \frac{p^3(s)}{2r^2 c} \frac{\partial}{\partial t} + \frac{p^2(s)}{rc^2} \frac{\partial^2}{\partial t^2} \right], \\
F_{ijkl}(s) &:= n_i n_j n_k n_l \left[\frac{35p^8(s)}{8r^4} + \frac{35p^7(s)}{8r^3 c} \frac{\partial}{\partial t} + \frac{15p^6(s)}{8r^2 c^2} \frac{\partial^2}{\partial t^2} - \frac{p^5(s)}{12rc^3} \frac{\partial^3}{\partial t^3} + \frac{p^4(s)}{24c^4} \frac{\partial^4}{\partial t^4} \right] \\
&\quad - n_i n_j \delta_{kl} \left[\frac{15p^6(s)}{4r^4} + \frac{15p^5(s)}{4r^3 c} \frac{\partial}{\partial t} + \frac{3p^4(s)}{2r^2 c^2} \frac{\partial^2}{\partial t^2} - \frac{p^3(s)}{12rc^3} \frac{\partial^3}{\partial t^3} \right] \\
&\quad - \delta_{ij} \delta_{kl} \left[\frac{3p^4(s)}{8r^4} + \frac{3p^3(s)}{8r^3 c} \frac{\partial}{\partial t} + \frac{p^2(s)}{8r^2 c^2} \frac{\partial^2}{\partial t^2} \right].
\end{aligned} \tag{23}$$

In what follows we drop the post Newtonian source terms that are quadratic in the perturbation fields. Note that these contributions would lead to corrections in the energy emission formula which are quadratic in G . Therefore we will neglect them and proceed with $T^{\mu\nu}$ as the main contribution to the source, i.e. $S = \eta_{\mu\nu} T^{\mu\nu}$. If we consider a perfect, non viscous fluid with mass density ρ , pressure \mathcal{P} and velocity field $\mathbf{v} = (v^1, v^2, v^3)$, we have

$$\begin{aligned}
T^{00}(t, \mathbf{x}) &= c^2 [\rho(t, \mathbf{x}) + \mathcal{O}(c^{-2})], \\
T^{0i}(t, \mathbf{x}) &= c [\rho(t, \mathbf{x}) v^i(t, \mathbf{x}) + \mathcal{O}(c^{-2})], \\
T^{ij}(t, \mathbf{x}) &= \rho(t, \mathbf{x}) v^i(t, \mathbf{x}) v^j(t, \mathbf{x}) + \mathcal{P}(t, \mathbf{x}) \delta_{ij} + \mathcal{O}(c^{-2}).
\end{aligned} \tag{24}$$

We express the spatial integrals over the source in (11) and (22) using the following momenta of the energy-momentum tensor,

$$M^{I_n}(t) := \frac{1}{c^2} \int_{\mathbb{R}^3} d^3x T^{00}(t, \mathbf{x}) x^{I_n}, \quad S^{ijI_n}(t) := \int_{\mathbb{R}^3} d^3x T^{ij}(t, \mathbf{x}) x^{I_n}, \tag{25}$$

and the quantities

$$\begin{aligned}
\mathcal{M}_{klm}^{I_n}(t) &:= \int_{\mathbb{R}} ds q(s) \frac{p^k(s)}{r^l c^m} \frac{\partial^m}{\partial t^m} M^{I_n}(\tau(t, s)), \\
\mathcal{S}_{klm}^{ijI_n}(t) &:= \int_{\mathbb{R}} ds q(s) \frac{p^k(s)}{r^l c^m} \frac{\partial^m}{\partial t^m} S^{ijI_n}(\tau(t, s)).
\end{aligned} \tag{26}$$

Here I_n denotes a string of indices $i_1 \dots i_n$, $n = 0, 1, 2 \dots$, and $x^{I_n} := x^{i_1} \dots x^{i_n}$. Moreover, we will denote $\mathcal{M} := \mathcal{M}_{000}$. It is useful to introduce the following linear combinations of

the quantities (26),

$$\begin{aligned}
D^i(t) &:= \mathcal{M}_{210}^i(t) + \mathcal{M}_{101}^i(t), \\
Q_1^{ij}(t) &:= \frac{3}{2}\mathcal{M}_{420}^{ij}(t) + \mathcal{M}_{311}^{ij}(t) + \frac{1}{2}\mathcal{M}_{202}^{ij}(t), \\
Q_2^{ij}(t) &:= -\mathcal{S}_{000}^{ij}(t) - \frac{1}{2}\mathcal{M}_{220}^{ij}(t) - \frac{1}{2}\mathcal{M}_{111}^{ij}(t), \\
O_1^{ijk}(s) &:= \frac{5}{2}\mathcal{M}_{630}^{ijk}(t) + \frac{1}{2}\mathcal{M}_{521}^{ijk}(t) + \frac{1}{2}\mathcal{M}_{412}^{ijk}(t) + \frac{1}{6}\mathcal{M}_{303}^{ijk}(t), \\
O_2^{ijk}(s) &:= -\mathcal{S}_{210}^{ijk}(t) - \mathcal{S}_{101}^{ijk}(t) - \frac{3}{2}\mathcal{M}_{430}^{ijk}(t) - \frac{1}{2}\mathcal{M}_{321}^{ijk}(t) - \mathcal{M}_{212}^{ijk}(t), \\
H_1^{ijkl}(s) &:= \frac{35}{8}\mathcal{M}_{840}^{ijkl}(t) + \frac{35}{8}\mathcal{M}_{731}^{ijkl}(t) + \frac{15}{8}\mathcal{M}_{622}^{ijkl}(t) - \frac{1}{12}\mathcal{M}_{513}^{ijkl}(t) + \frac{1}{24}\mathcal{M}_{404}^{ijkl}(t), \\
H_2^{ijkl}(s) &:= -\mathcal{S}_{420}^{ijkl}(t) - \mathcal{S}_{311}^{ijkl}(t) - \mathcal{S}_{202}^{ijkl}(t) \\
&\quad - \frac{15}{4}\mathcal{M}_{640}^{ijkl}(t) - \frac{15}{4}\mathcal{M}_{531}^{ijkl}(t) - \frac{3}{2}\mathcal{M}_{422}^{ijkl}(t) + \frac{1}{12}\mathcal{M}_{313}^{ijkl}(t), \\
H_3^{ijkl}(s) &:= -\mathcal{S}_{220}^{ijkl}(t) - \mathcal{S}_{111}^{ijkl}(t) - \frac{3}{8}\mathcal{M}_{440}^{ijkl}(t) - \frac{3}{8}\mathcal{M}_{331}^{ijkl}(t) - \frac{1}{8}\mathcal{M}_{222}^{ijkl}(t).
\end{aligned} \tag{27}$$

Taking into account up to quadrupoles, and hexadecapoles, for γ^{ij} , and φ , respectively, the asymptotic fields (11) and (22) can be written in terms of the quantities (25), (26) and (27)

$$\gamma^{ij}(t, \mathbf{x}) = \frac{2G}{c^4 r} \frac{\partial^2}{\partial t^2} M^{ij}(t - r/c), \tag{28}$$

$$\begin{aligned}
\varphi(t, \mathbf{x}) &= -\frac{G}{3a c^2 r} \left[\mathcal{M}(t) + n_i D^i(t) + n_i n_j Q_1^{ij}(t) + \delta_{ij} Q_2^{ij}(t) \right. \\
&\quad + n_i n_j n_k O_1^{ijk}(t) + n_i \delta_{jk} O_2^{ijk}(t) \\
&\quad \left. + n_i n_j n_k n_l H_1^{ijkl}(t) + n_i n_j \delta_{kl} H_2^{ijkl}(t) + \delta_{ij} \delta_{kl} H_3^{ijkl}(t) \right], \tag{29}
\end{aligned}$$

where we have used the same symbols γ^{ij} and φ for the approximations. From (26) we infer that the dimensionless field $a\varphi$ depends on a through the function p . This irrational dependence on r arises by the same reason as the Yukawa like terms $e^{-r/\sqrt{6a}}$ in the solutions for φ for an isolated system, since the field φ has a range $1/\sqrt{6a}$ as per equation (6). In the same way as in the $1/c$ expansion of the metric field [12, 13], the approximation is parametrised by a dimensionless parameter a/ℓ^2 , where ℓ is a typical length depending on the model. As for the $1/c$ expansion of the field of a quasi static isolated source, also in the present case ℓ corresponds to the distance from the source r , i.e. the distance the wave has propagated through. Consequently, even the field $ra\varphi$ converges to zero for $r \rightarrow \infty$, that is, other than the pure metric field (28), the scalar field decays faster than $1/r$. This

is expected since the scalar field is massive, leading to a nontrivial dispersion relation for a propagating wave.

IV. THE ENERGY–MOMENTUM COMPLEX

In order to derive the energy flux of a gravitational wave in $f(R)$ inspired scalar tensor theory, we need an analogue to the Landau–Lifshitz complex $t_{LL}^{\mu\nu}$ in GR [15]. This can be obtained by using the method in [16], where an energy-momentum complex in the Brans–Dicke theory is presented. This method applies to theories derived from convex or concave functions f , i.e. theories that are equivalent to scalar–tensor theories. In this section, we consider a general metric tensor $g_{\mu\nu}$ and scalar field ϕ . In particular, we do not constrain the asymptotic values of these fields. For an alternative derivation of the energy–momentum complex cf. [9, 17].

Defining

$$X^{\mu\nu} := R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \frac{1}{8a\phi}(\phi - 1)^2g^{\mu\nu} - \frac{1}{\phi}(\nabla^\mu\nabla^\nu\phi - g^{\mu\nu}\square_g\phi) \quad (30)$$

and

$$U^{\mu\nu\lambda\sigma} := \phi^2(-g)(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\lambda\nu}), \quad (31)$$

we can write the generalisation of the Landau–Lifshitz complex as

$$t^{\mu\nu} = \frac{c^4\phi}{8\pi G} \left[\frac{1}{2\phi^2(-g)}\partial_\lambda\partial_\sigma U^{\mu\lambda\nu\sigma} - X^{\mu\nu} \right]. \quad (32)$$

By construction, the energy–momentum conservation laws then can be cast into the form

$$\partial_\mu [\phi(-g)(T^{\mu\nu} + t^{\mu\nu})] = 0. \quad (33)$$

Using (30) and (31), it is straight forward to express the energy momentum complex (32) in terms of the fields $g_{\mu\nu}$, ϕ , their first partial derivatives and the connection coefficients $\Gamma_{\mu\nu}^\lambda$ as

$$\begin{aligned} t^{\mu\nu} = & \phi t_{LL}^{\mu\nu} + \frac{c^4}{16\pi G\phi} \left[2g^{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi - 2\partial^\mu\phi\partial^\nu\phi - \frac{1}{4a}(\phi - 1)^2g^{\mu\nu} \right] \\ & + \frac{c^4}{8\pi G} \left[g^{\mu\nu} \left(2\partial^\lambda\phi\Gamma_{\lambda\sigma}^\sigma - \partial_\lambda\phi\Gamma_{\sigma\rho}^\lambda g^{\sigma\rho} \right) + g^{\mu\lambda}g^{\nu\sigma}\partial_\rho\phi\Gamma_{\lambda\sigma}^\rho \right. \\ & + \left(\partial^\mu\phi\Gamma_{\lambda\sigma}^\nu + \partial^\nu\phi\Gamma_{\lambda\sigma}^\mu \right) g^{\lambda\sigma} - \left(\partial^\mu\phi g^{\nu\lambda} + \partial^\nu\phi g^{\mu\lambda} \right) \Gamma_{\lambda\sigma}^\sigma \\ & \left. - \left(g^{\mu\lambda}\Gamma_{\lambda\sigma}^\nu + g^{\nu\lambda}\Gamma_{\lambda\sigma}^\mu \right) \partial^\sigma\phi \right]. \end{aligned} \quad (34)$$

The energy flux in an arbitrary direction x^i is given by the component t^{0i} .

V. ENERGY EMISSION OF ISOLATED SYSTEMS

Consider a plane gravitational wave propagating in vacuum in the x^i direction. In addition to the gauge (9), it is possible to perform a further gauge transformation which makes the φ independent part of the perturbation transverse and traceless (TT) [2, 6, 9], such that we can write

$$h^{\mu\nu}(t - x^i/c) = \gamma_{TTi}^{\mu\nu}(t - x^i/c) - 2a\eta^{\mu\nu}\varphi(t, x^i) \quad (35)$$

In this gauge, we evaluate the energy flux to leading order in the perturbation fields $\gamma_{\mu\nu}$ and φ . By angle brackets we denote the average over a four-dimensional spacetime region with an extension which is much larger than a typical wavelength. The formula (34) then yields

$$t^{0i} = \frac{c^4}{32\pi G} \left\langle \partial_0 \gamma_{TTi}^{jk} \partial_0 \gamma_{TTi}^{jk} + 24a^2(\partial_0\varphi)^2 \right\rangle. \quad (36)$$

The first term on the right hand side of (36) can be evaluated in the same way as in GR by means of (28) and the trace-free quadrupole tensor

$$\mathcal{Q}^{ij}(t) = \frac{1}{c^2} (3M^{ij} - \delta_{ij}M^{kk}), \quad (37)$$

From (29) we obtain for the second term on the right hand side of (36) the asymptotic value

$$\begin{aligned} 24a^2(\partial_0\varphi)^2 = & \frac{8G^2}{3c^6r^2} \left[\dot{\mathcal{M}}^2 + 2n_i\dot{\mathcal{M}}\dot{D}^i + n_i n_j \left(2\dot{\mathcal{M}}\dot{Q}_1^{ij} + \dot{D}^i\dot{D}^j \right) + 2\delta_{ij}\dot{\mathcal{M}}\dot{Q}_2^{ij} \right. \\ & + 2n_i n_j n_k \left(\dot{\mathcal{M}}\dot{O}_1^{ijk} + \dot{D}^i\dot{Q}_1^{jk} \right) - 2n_i\delta_{jk} \left(\dot{\mathcal{M}}\dot{O}_2^{ijk} + \dot{D}^i\dot{Q}_2^{jk} \right) \\ & + n_i n_j n_k n_l \left(2\dot{\mathcal{M}}\dot{H}_1^{ijkl} + 2\dot{D}^i\dot{O}_1^{jkl} + \dot{Q}_1^{ij}\dot{Q}_1^{kl} \right) \\ & + 2n_i n_j \delta_{kl} \left(\dot{\mathcal{M}}\dot{H}_2^{ijkl} + \dot{D}^i\dot{O}_2^{jkl} + \dot{Q}_1^{ij}\dot{Q}_2^{kl} \right) \\ & \left. + \delta_{ij}\delta_{kl} \left(2\dot{\mathcal{M}}\dot{H}_3^{ijkl} + \dot{Q}_2^{ij}\dot{Q}_2^{kl} \right) \right]. \end{aligned} \quad (38)$$

The total power of the source is obtained by integrating (36) over a sphere with radius r :

$$\begin{aligned} P = & \frac{G}{3c} \left\langle \dot{\mathcal{M}}^2 + \frac{1}{3} \left[\dot{\mathcal{M}} \left(2\dot{Q}_1^{ij} + 6\dot{Q}_2^{ij} \right) + \dot{D}^i\dot{D}^j \right] \right. \\ & + \frac{1}{15} \left[\ddot{\mathcal{Q}}^{ij}\ddot{\mathcal{Q}}^{ij} + \dot{\mathcal{M}} \left(6\dot{H}_1^{iijj} + 10\dot{H}_2^{iijj} + 30\dot{H}_3^{iijj} \right) + \dot{D}^i \left(6\dot{O}_1^{iijj} + 10\dot{O}_2^{iijj} \right) \right. \\ & \left. \left. + \dot{Q}_1^{ii} \left(\dot{Q}_1^{jj} - 10\dot{Q}_2^{jj} \right) + 2\dot{Q}_1^{ij}\dot{Q}_1^{ij} + 15\dot{Q}_2^{ij}\dot{Q}_2^{ij} \right] \right\rangle, \end{aligned} \quad (39)$$

where the dot denotes the derivative with respect to t . Note that the additional degree of freedom represented by the scalar field predicts radiation of all multipoles, in particular monopoles and dipoles. These are of lower order than the original quadrupole contribution in GR represented by the moments \mathcal{Q}^{ij} , and could principally lead to a non negligible contribution in concrete applications. Since the quantities \mathcal{M}^I and \mathcal{S}^I as well as their derivatives vanish in the limit $r \rightarrow \infty$, the energy radiated by means of the scalar degree of freedom is absorbed as the field φ disperses completely when propagating to timelike infinity.

VI. APPLICATIONS

The most interesting application of the formal result (39) is the energy loss of binary systems by the emission of gravitational radiation, in particular binary pulsars such as the PSR J0737-3039 system [18, 19]. In quadratic $f(R)$ gravity, the non-relativistic motion of compact objects is governed by the Newtonian potential with an additional Yukawa correction. This implies that the Keplerian orbits also need appropriate corrections, cf. for example [12]. A general treatment including generic orbits is beyond the scope of this work. In order to obtain a rough estimate for the correction terms in (39), we apply the formula to the radiation of binaries in circular orbits. For this application we expect that the total mass of the system changes on a time scale which is much larger than the orbital period. Hence, the monopole contribution to (39) is negligible, $\dot{\mathcal{M}} \approx 0$, whereas for appropriate mass densities the dipole term is the leading order contribution.

Consider a binary system consisting of two point masses m_1 and m_2 in a circular orbit moving at angular velocity ω . For $m_1 \neq m_2$, the dipole moment does not vanish. The only non-relativistic correction is a modification of Kepler's third law [12],

$$\omega^2 = \frac{G(m_1 + m_2)}{d^3} \left[1 + \frac{1}{3} \left(1 + \frac{d}{\sqrt{6}a} \right) e^{-d/\sqrt{6}a} \right], \quad (40)$$

which can be numerically solved for the orbital separation d . This non-relativistic correction is parametrised by a/d^2 , while the corrections arising from the propagation of the field φ are a function of a/r^2 . The choice of the ratio d/r is constrained to be small for the multipole expansion to be viable.

We choose coordinates such that the motion is restricted to the (x^1, x^2) -plane. In a

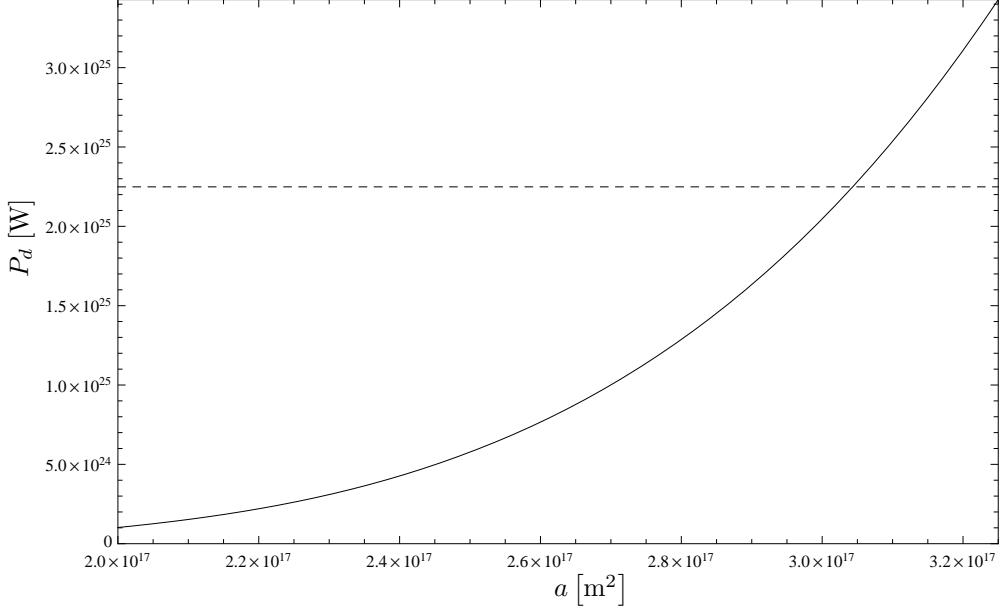


FIG. 1: Dipole power P_d as a function of the model parameter a for the data (45). The value of the quadrupole power as predicted by GR, $P_q^{\text{GR}} = 2.248 \cdot 10^{25}$ W, is indicated by the dashed line.

corotating frame, the mass density can be written as

$$\rho(\mathbf{x}) = \delta(x^2)\delta(x^3) [m_1\delta(x^2 - d/2) + m_2\delta(x^2 + d/2)]. \quad (41)$$

The dipole moments are therefore given by

$$M^1(t) = \frac{d(m_1 - m_2) \cos(\omega t)}{2}, \quad M^2(t) = M^1\left(t - \frac{\pi}{2\omega}\right), \quad M^3(t) = 0, \quad (42)$$

leading to

$$\begin{aligned} \mathcal{M}_{klm}^1(t) &= \frac{d(m_1 - m_2)}{2} \int_{\mathbb{R}} ds \frac{p^k(s)}{r^l c^m} q(s) \frac{\partial^m}{\partial t^m} \cos(\omega\tau(t, s)), \\ \mathcal{M}_{klm}^2(t) &= \mathcal{M}_{klm}^1\left(t - \frac{\pi}{2\omega}\right). \end{aligned} \quad (43)$$

Taking the average of the dipole contribution over one period $\mathcal{T} := 2\pi/\omega$ we obtain

$$\begin{aligned} P_d &= \frac{G}{9c} \left\langle \left(\dot{D}^1\right)^2 + \left(\dot{D}^2\right)^2 \right\rangle \\ &= \frac{G\omega}{9\pi c} \int_0^{\mathcal{T}} dt \left[\left(\dot{\mathcal{M}}_{210}^1(t)\right)^2 + \left(\dot{\mathcal{M}}_{101}^1(t)\right)^2 + 2\dot{\mathcal{M}}_{210}^1(t)\dot{\mathcal{M}}_{101}^1(t) \right]. \end{aligned} \quad (44)$$

In Figure 1 we plot P_d against the model parameter a for the data

$$\begin{aligned} m_1 &= 1.337 M_{\odot}, & m_2 &= 1.250 M_{\odot}, \\ \omega &= 0.7112 \cdot 10^{-4} \text{ s}^{-1}, & r &= 10^{10} \text{ m}, \end{aligned} \quad (45)$$

which is inspired by the PSR J0737-3039 system, but moving in a circular orbit with angular velocity $2\pi/\mathcal{T}$, where $\mathcal{T} = 8.834 \cdot 10^3$ s is the measured orbital period [18]. The parameter a lies within the interval $\mathcal{I} = [2 \cdot 10^{17} \text{ m}^2, 3.25 \cdot 10^{17} \text{ m}^2]$, such that the dipole term is comparable to the value of the quadrupole power as predicted by GR, $P_q^{\text{GR}} = 2.248 \cdot 10^{25} \text{ W}$. From (40) we infer that $d \approx 9.5 \cdot 10^8 \text{ m}$ for $a \in \mathcal{I}$; the Newtonian value is $d = 8.798 \cdot 10^8 \text{ m}$. We choose the distance r from the source such that $d/r \approx 1/10$. For this data, the dipole power equals 1% of the GR value at $a \approx 1.7 \cdot 10^{17} \text{ m}^2$.

VII. CONCLUSION

We have derived the $f(R)$ correction terms to the GR quadrupole formula for the emission of gravitational radiation to leading order. An important result is that, in contrast to GR, quadratic $f(R)$ theory predicts the radiation of monopoles and dipoles. This is the case for nearly every alternative metric gravity theory known today [1].

When considering a hypothetical binary similar to the double pulsar PSR J0737-3039, we found for the constant a an upper limit of about $1.7 \cdot 10^{17} \text{ m}^2$. It would be interesting to study the result for real data of pulsars to see whether this limit can be improved. Notice that the limit on a we got from the geodetic precession using the Gravity Probe B data and the precession of the pulsar PSR J0737-3039B is somewhat more stringent [12]. When completing our paper we came aware of the recent publication [9] as well as the preprint [20], which contain equivalent results to some extent. The multipole expansion we focus on in our paper is, however, complementary to [9], and the treatment in [20] is based on different assumptions concerning the conservation laws.

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